PHASE SHIFT AMBIGUITIES FOR SPIN-INDEPENDENT SCATTERING

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The representation of scattering data by phase shifts is by now a deeply-rooted tradition accepted universally by each new generation of physicists. After measuring a differential cross-section,

$$(1) \qquad d\sigma / d\Omega = |f(\theta)|^2$$

at a certain energy (here it is assumed that there is no spin dependence), one seeks to represent the data by means of a set of real angles such that

$$(2) \qquad f(\theta) = (1/k) \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\theta),$$

where k is the momentum in the center-of-mass system[1]. The advantage of such a representation is in situations in which one expects and finds that only a small number of non-vanishing phase shifts will suffice. It is an implicit assumption, never proved, that there will be a unique set of such phase shifts, aside from well-known and obvious ambiguities. (We exclude here a consideration of experimental uncertainties.) We

show that this assumption is false by exhibiting a counter-example.

The problem here is that specification of the phase shifts also specifies the phase of the scattering amplitude f, whereas only its absolute value is determined experimentally. Thus we inquire if it is possible for two or more different sets of phase shifts to satisfy the experimental data. The ambiguities which we discuss here for spin-independent scattering are somewhat analogous to the famous Fermi-Yang ambiguities encountered in the early history of π p scattering.

It is well known that phase shifts are determined only up to modulo π . Furthermore, if the signs of all the phase shifts are changed, the scattering amplitude is changed to its negative complex conjugate; thus its phase is changed, but its absolute value is unchanged. These ambiguities we call the trivial ambiguities.

There are few statements in the literature about the problem of uniqueness in spin-independent scattering. Klepikov[2] seems to indicate that the only ambiguities are of the trivial kind, which can be resolved by use of dispersion relations. (Klepikov's concern, however, is with extracting unique phase shifts from experimental data.) Wu and Ohmura[3] point out that the unitarity equation provides a nonlinear integral equation for the phase of the scattering amplitude, given its absolute value. Our counter-example shows that this integral equation need not have a unique solution.

One readily sees that for S waves only, and for S and P waves only, the phase shifts are determined uniquely except for the trivial

ambiguities. For the case when the only non-vanishing phase shifts are for S, P, and D waves, the phase shifts are <u>not</u> unique. Consider, for example, a differential cross-section

$$(3)\quad |kf(\theta)|^2 = 2.1606\ P_0(\cos\theta) + 0.2732\ P_1(\cos\theta) + 2.6879\ P_2(\cos\theta) - 1.8924\ P_3(\cos\theta) + 1.5040\ P_4(\cos\theta).$$

This cross-section can be represented by two different sets of phase shifts, either

$$(4)\quad \delta_0 = -23°20', \qquad \delta_1 = -43°27', \qquad \delta_2 = 20°,$$

or

$$(4')\quad \delta_0' = 98°51', \qquad \delta_1' = -26°33', \qquad \delta_2' = 20°.$$

Tha absolute value, phase, and real and imaginary parts of $kf(\theta)$ are shown in Figs. 1 - 3.

By crude search methods, we have found all of the ambiguities for the case of non-vanishing S, P, and D waves only. They can be characterized as follows: For each value of δ_2, such that

$$12°31,8' \leq \delta_2 \leq 24°09.2',$$

there exists one and only one case of an ambiguity. That is to say, for each such δ_2, there exists two phase shifts, δ_0 and δ_1, such that a different set of phase shifts, $\{ \delta_0', \delta_1', \delta_2 \}$, exists which gives the

same $|kf(\theta)|$. The different phase shifts are related by

$$\delta_1 + \delta_1' = \pi/2 + \delta_2, \tag{5}$$

$$\delta_0 + \delta_0' = \tan^{-1}\{\sin\delta_2 \cos \delta_2/[\, 1/5 - \sin^2\delta_2]\}, \tag{6}$$

$$\sin^2\delta_0 + 3 \sin^2\delta_1 = \sin^2\delta_0' + 3 \sin^2\delta_1' , \tag{7}$$

$$\sin^2\delta_1 + \sin\delta_1 \sin(2\delta_0-\delta_1) = \sin^2\delta_1' + \sin\delta_1' \sin(2\delta_0'-\delta_1'). \tag{8}$$

Equations (5)-(8) are obtained by expanding $|kf|^2$ in powers of cos θ in terms of the set$\{\delta_\ell'\}$ and equating it to $|kf|^2$ expressed in terms of the set$\{\delta_\ell'\}$. The highest-order non-vanishing phase shift is uniquely determined. Suppose that there are L + 1 non-vanishing phase shifts. WitH δ_L determined, there remain 2L equations and 2L unknowns (L unprimed and L primed phase shifts). Thus it is certainly not unreasonable that the ambiguities exist.

Because of the crude mathematical methods available it is an extremely difficult task to search for ambiguities present with a larger number of non-vanishing phase shifts. For example, when one includes F waves, there are five equations, nonlinear in trigonometric functions of five unknowns. This being the case we have attempted to embroider on the three-phase shift ambiguities. For the case of the three lowest order even-parity waves, S, D, and G, not vanishing, the same type of ambiguity exists. It also exists for non-vanishing S, P, and D waves in two- and four-dimensional space.

We have shown two things in finding these ambiguities:

(1) The assumption that there always exists a unique phase-shift representation for scattering data is false.

(2) The ambiguities which undercut this assumption are not very extensive, at least for small numbers of non-vanishing phase shifts, so that the assumption is good enough for practical applications.

Of course, these ambiguities for a few phase shifts could be resolved either by using interference with coulomb scattering or by changing the energy. Thus there seems to be no experimental importance attached to these ambiguities. Nevertheless the problem of uniqueness for an arbitrary number of non-vanishing phase shifts in the analysis of spin-independent scattering data remains unsolved. Also it may be that the type of ambiguity we have found here is present in spin-dependent scattering analysis as well, in addition to the well-known ambiguities.[4]

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References:

(1) See, for example, Marvin L. Goldberger and Kenneth M. Watson, Collision Theory, (John Wiley & Sons, Inc., New York, 1964), page 226.

(2) N. P. Klepikov, Soviet Physics JETP 14, 846 (1962).

(3) T. Wu and T. Ohmura, Quantum Theory of Scattering, (Prentice-Hall, inc., Englewood Cliffs, New Jersey, 1962), page 100.

(4) R. van Wageningen, Ann. Physics 31, 148 (1965).

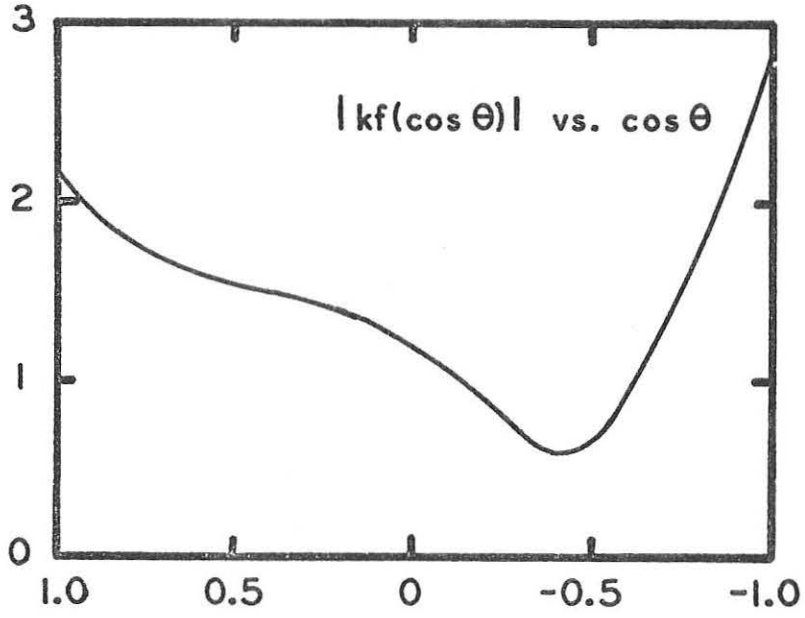


Fig. 1 The absolute value of kf given by the phase shifts, Eq. (4) and Eq. (4').

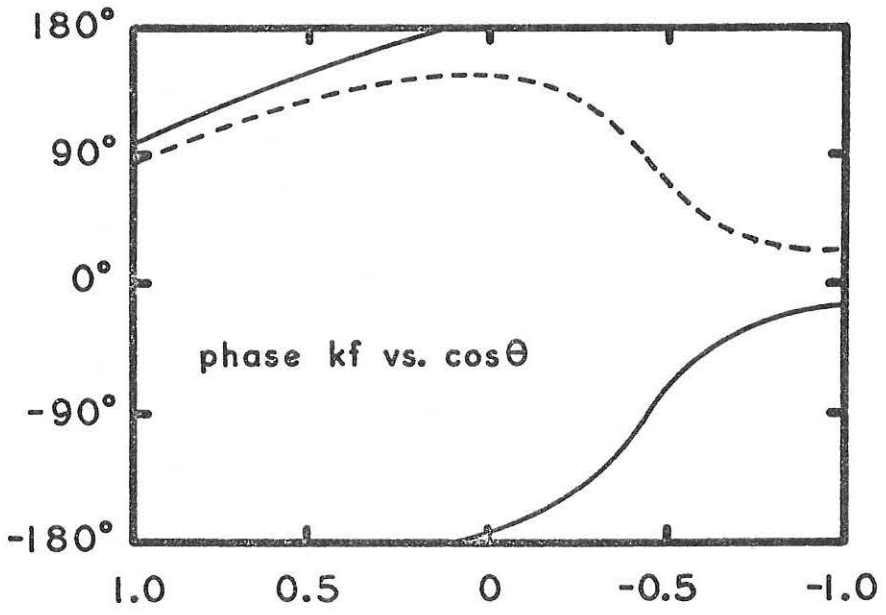


Fig. 2 The phase of kf given by the phase shifts, Eq. (4), (solid curve) and by the phase shifts, Eq. (4'), (dashed curve).

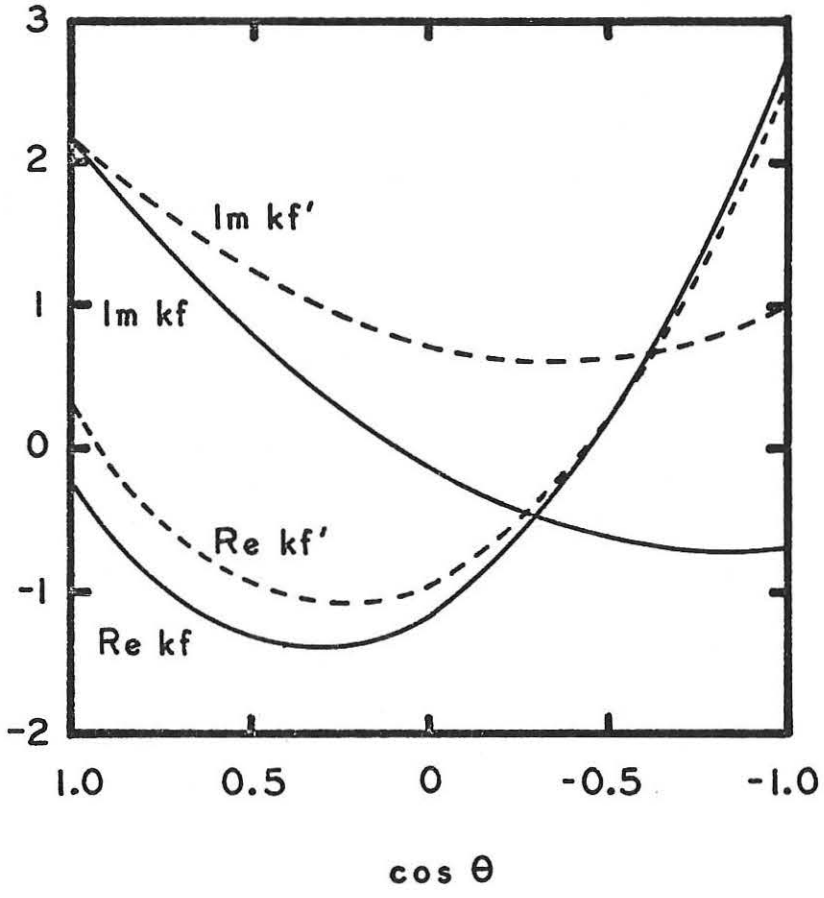


Fig. 3 The real and imaginary parts of kf given by the phase shifts, Eq. (4), (solid curve) and by the phase shifts, Eq. (4'), (dashed curve).